MPP Calculus

Take-Home Exam

Please drop off the completed exam in my mailbox in Fisher Hall (CHAN) by Friday at 4:30 pm.

1. Continuity

(a) Suppose that *f* assumes only rational values over an interval; moreover, it assumes two distinct values over the interval. Prove that *f* must be a discontinuous function.

(b) Find the points of discontinuity of the following functions:

1. $f(x) = \frac{1-\log x}{1+\log x}$
2. $f(x) = \frac{1+e^{-x}}{e^x+\log(1+x)}$
3. $f(x) = \frac{3x^2+\int t \log t dt}{(1+x)\int_a^x \frac{1}{t} dt}$

2. Rolle's Theorem

Suppose that *f* is defined everywhere and that all its derivatives are continuous, in particular, $f''' < 0$. Show that *f* has at most two critical points.

3. Graphing

Sketch the graphs of the following two functions and explicitly label all the points of interest (minima, maxima, points of intersection, roots, etc.):

$$f(x) = \frac{x}{1+x} \qquad (1)$$

$$f(x) = \frac{e^{-x}}{1+e^x} \qquad (2)$$

4. Inverse functions

Find the inverse functions for the following two functions:

$$f(x) = \sqrt[3]{1+\log x} \qquad (3)$$

$$f(x) = 1/x \qquad (4)$$

Also, be sure to state the domains for *f* and f^{-1}.

5. Differentiation

Find the first and second derivatives of the following two functions:

$$f(x) = \int_{x^2+2}^{0} \sqrt{1+t^2}\, dt \tag{5}$$

$$f(x) = x[1 + (\log(1+x))^3] \tag{6}$$

6. Integration

Solve the following two integrals:

$$\int_0^1 x e^{-x}\, dx \tag{7}$$

$$\int_e^{\infty} \frac{1}{x \log x}\, dx \tag{8}$$

7. Taylor polynomials

Find the Taylor polynomials of degree 4 for the following functions:

1. $f(x) = \frac{1}{1-x}$ about a = 2:
2. $f(x) = \exp(e^x)$ about a = 0:

8. Sequences & series

(a) Show that the sequence $\{1 + 1/n^3\}$ converges to 1 directly from the definition of convergence.

(b) Find the sum of the sequences: (i) $\{1/n\}$ and (ii) $\{1/n^3\}$

(c) Find the following limit: $\lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n$

9. Matrix algebra

Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) Solve the following matrix operations (if they are indeed valid):

1. **AB**
2. **B + C**
3. **BAC**
4. (**BAC**)$^{-1}$
5. det(**BAC**)
6. **BA**^{T}

(b) Use Cramer's Rule to solve the following system

$x_1 + 2x_2 + 3x_3 = 4$

$2x_1 + 3x_2 - x_3 = 1$

$x_1 - 2x_2 + 4x_3 = -2$

10. Optimisation with constraints

Solve the following optimisation problems:

1. $\max_{x_1,x_2} U(x_1,x_2) = x_1^2 x_2^2$ subject to the restriction that $x_1 + \frac{3}{2}x_2 = 100$.
2. $\min_{K,L} rK + wL$ subject to the restriction that $K^{1/3}L^{2/3} = 100$.
3. Repeat question 2 with the modification $K^{1/3}L^{2/3} \geq 100$ and $K \geq 0, L \geq 1000$.
4. $\min_{x_1,x_2} 4x_1 + x_2$ subject to the restrictions

$$x_1^2 - 4x_1 + x_2 \geq 1$$

$$-2x_1 - 3x_2 \geq -11$$

$$x_1, x_2 \geq 0$$