MPP Calculus Problem Set 3

Due Thursday, August 29

1. Integration

• Note that if $g = x^2 + C$ then $f \equiv g' = 2x$. Thus $\int f = x^2 + C$.That is, whenever we are integrating there is an implicit "constant of integration". However, we can ignore this nuisance if we consider the definite integral since$\int_a^b f = [x^2 + C]_a^b = [b^2 + C] - [a^2 + C] = b^2 - a^2$. But you should always understand that integration and differentiation differ by a constant.

• Spivak Chapter 13: 7 (even), 8 (odd), 12, 33.

• Solve the following integrals by substitution (i.e. let $w = h(x)$ and so $dw = h'(x)dx$). (Hint: $a^{-1} = 1/a$ for all $a \neq 0$ and $1 \cdot f = f$ for all f.)

1. $\int x\sqrt{1 - x^2}\, dx$
2. $\int \exp(e^x) e^x dx$
3. $\int x^2 e^{-x^3}\, dx$
4. $\int \log(\log x)[x \log x\,]^{-1}\, dx$

• Solve the following integrals by parts (i.e. $\int udv = uv - \int vdu$).

1. $\int x \log x dx$
2. $\int x^3\, e^x dx$
3. $\int x^2\, e^{-x} dx$

2. Fundamental theorem of calculus

Recall: First FTC states if $F(x) = \int_a^{g(x)} f(t)dt$ then $F'(x) = f\big(g(x)\big)g'(x)$. And the Second FTC states if $g' = f$ then $\int_a^b f = g(b) - g(a)$.

• Spivak Chapter 14: 11, 21.

• Find the derivatives of the following functions

1. $F(x) = \int_{10}^{x} \{1 + \exp[t^2 - 1]\}\, dt$
2. $F(x) = \int_{3x^2}^{15} (\log t - f(t))\, dt$
3. $F(x) = \int_a^{\log(1+x^2)} f(w)dw$
4. $F(x) = \int_{g(x^2)}^{500} zh(z)dz$

3. Taylor polynomials

Recall Taylor's theorem states that any function can be approximated by a polynomial:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$$

• Find the fourth order Taylor polynomials for the following functions about the point *a* = 1:

1. $f(x) = (1+x)^{-1}$
2. $f(x) = \exp(x)$
3. $f(x) = \exp(e^x)$

• Find the approximate values of $\log(3)$ and e^2 without using a calculator.

4. Sequences

• Spivak Chapter 22: 3, 27 (b).

• Show that the sequence $\{1/n^2\}$ converges to zero directly from the definition of convergence. (Recall that a sequence $\{a_n\}$ is said to converge to 1 (i.e. $\lim_{n\to\infty} a_n = 1$) if for every $\varepsilon > 0$ there exists $M \in \boldsymbol{N}$ such that whenever $n > M$ then $|a_n - 1| < \varepsilon$.)

• Does the sequence $\left\{\frac{\log n}{n}\right\}$ converge or diverge?

5. Miscellaneous

• Show that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

• Find the derivative of $F(x) = \int_3^{\int_1^x \log t\,dt} \frac{1}{1+\exp(1+t^2)}\,dt$

• Suppose f is defined everywhere and that all its derivatives are continuous, in particular, $f''' > 0$. Show that f has at most two critical points.

• Sketch the graph of the function $f(x) = x^2 - \frac{1}{x}$.