MPP Calculus Problem Set 1

1. Basic properties of numbers

Spivak Chapter 1 exercises: 4 (odd), 11 (even), 17 (all), and 25.

2. Numbers of various sorts

2.1. Prove using mathematical induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad (1)$$

Hint: Recall that you need to verify $P(1)$ and then you assume that the formula is true for an arbitrary $k \in$ $\boldsymbol{N}$, and then you need to verify the truth of the formula for the case of $k + 1$.

2.2. Show that $\sqrt{2} + \sqrt{2}$ is irrational.

* Educational: Suppose that there are finitely many prime numbers (from which we will get a contradiction). That is, suppose that are M prime numbers (where M is finite) and let $P_1, P_2, \ldots, P_M$ represent all the prime numbers in existence. Now define a new number

$$P^* = P_1 \cdot P_2 \cdot \ldots \cdot P_M + 1 \qquad (2)$$

If P^* is prime then this contradicts the conjecture that there are only M prime numbers. Moreover, if P^* is not prime, then by definition we can decompose it into its prime factors; however, none of P_1 through P_M can divide P^* (since they would leave a remainder of 1), and thus there must be some prime number not included in the collection $\{P_1, \ldots, P_M\}$ that divides P^*.

3. Functions

Spivak Chapter 3 exercises: 3 (all).

4. Graphs

Spivak Chapter 4 exercises: 3 (odd), 4 (all), 17 (odd).

5. Limits

Spivak Chapter 5 exercises: 1 (all), 2 (all), 39 ((i), (v), (vi), (vii)).

Find the following limits involving the log and exp functions:

5.1.1. $\lim_{x\to 1} \exp(x) / [1 - \exp(x)]$

5.1.2. $\lim_{x\to 0} \log(x + 1) / \log[1 + \exp(x)]$

5.1.3. $\lim_{x\to 0} \frac{[x^n - \exp(x) + \log(x^n)]}{\{exp[x^2+1] - log[x^2+1]\}}$

5.1.4. $\lim_{x \to 0} \left\{\frac{3x+5}{x+1}\right\}^x$ Hint: note the title of these exercises.

5.1.5. $\lim_{x \to 1}\{log[5+3x^2] - log[a+x^2] + exp[-2x^2]exp(9x)\}$

Verify if the following limits exist. (Hint: If the limit of a function exists what do we know about its limiting value from either side?)

5.2.1. $\lim_{x \to 0} \frac{1}{[3+2^{1/x}]}$

5.2.2. $\lim_{x \to 0} \frac{[1+2^{1/x}]}{[3+2^{1/x}]}$

5.2.3. Using the δ- ε method of limits show that

$$\lim_{x \to 2} x^2 - 4 = 0 \qquad (3)$$

Hint: If the limit as x approaches 2 of $x^2 - 4$ is really zero, then for every $\delta > 0$ we can find $\varepsilon > 0$ such that whenever $0 < |x - 2| < \delta$ then $|(x^2 - 4) - 0| < \varepsilon$. Moreover, we can factor $x^2 - 4$, and to ensure that ε is not too large assume $\delta < 1$.

6. Continuity

Spivak Chapter 7 exercises: 1 (odd), 10.

6.1. Show that f (x) = |x| is continuous everywhere.

6.2. Find the discontinuities of the following functions:

1. $f(x) = 1/x$
2. $f(x) = (x-1)/[(x+3)(x-2)]$
3. $f(x) = (x+2)(x-1)/(x-3)^2$

7. Least upper bounds

Spivak Chapter 8 exercises: 1 (even).