# Introductory Statistics
# Winter 2004

**SUGGESTED SOLUTIONS TO MIDTERM EXAM**
March 2, 2004

*You have 75 minutes to answer the following questions. Each of the first five questions is worth 15 points. The five remaining questions are worth 5 points apiece. There is one bonus question at the end worth 10 points. The test is thus out of 100 with 10 possible bonus points. Allocate your time accordingly. You must show your work! Good Luck!*

Q1. (15 points) The makers of "Weight Gain 4000" have decided to market their product through the medium of infomercials. A call centre is set up to handle incoming calls from wannabe "beefcakes". Suppose that the number of phone calls coming in to the call centre (per hour) is described by a Poisson distribution with parameter $\mu = 16$. That is

$$p(x) = \frac{e^{-16}16^x}{x!} \quad x = 0,1,2,3,\ldots$$

(a) What is the probability that the call centre will receive exactly one phone call in the next hour?

*P(1) =* $\frac{e^{-16}16}{1}$*. You need not waste your time plugging away at your calculator to get a decimal value. However, if you did compute this, you would get something very close to zero.*

(b) What is the probability that the call centre will receive at least four calls in the next hour?

*$P(X \geq 4) = 1 - P(X < 4) = 1 - [p(0) + p(1) + p(2) + p(3)]$; where for each $p(\bullet)$ we just plug in the corresponding number into the formula for the Poisson.*

(c) Using Chebyshev's theorem, find an upper bound for the probability that the call centre will receive 28 or more calls in the next hour. (Hint: If $X$ is distributed Poisson with parameter $\mu$, then $\text{Var}(X) = \mu$.)

*Since the variance is equal to $\mu = 16$, this means its standard deviation is 4. By Chebyshev's theorem we know that at least $1 - 1/3^2 = 8/9$ of the information is contained within 3 standard deviations of the mean (which is 16 in this example). So at most 1/9 of the observation of X fall outside this range. Moreover, since there are there are two "sides" of the mean, the probability is even lower (since we are looking at the right tail).*

Q2. (15 points) Consider two mutual funds, "Howells Investments" (*X*) and "Riveras Funds" (*Y*), whose returns are both normally distributed: $X \sim N(10,5^2)$ and $Y \sim N(15,8^2)$.

(a) What is the probability that Howells Investment earns at least 15 percent? Likewise, what is the probability that Riveras Funds earns at least 15 percent?

*These calculations don't require you to look up your standard normal table. Just note that for Howells Investment a return of 15 percent is exactly one standard deviation from it mean, and so P(X > 15) is approximately 0.16. For Riveras Fund P(Y > 15) is exactly 0.5.*

(b) If an investor is very nervous about earning a negative return and has to choose between the two funds, which one should she pick?

*No need to do any calculations. Just note that to get a negative return we need to be two standard deviations away from the mean of X, while it is less than two standard deviations from the mean of Y, so the probability of a loss is greater with Y.*

(c) If an investor placed 1/3 of her money in Howells Investments and the remainder of her money in Riveras Funds, what is the expected return on her investment?

*E((1/3)X + (2/3)Y) = (1/3)E(X) + (2/3)E(Y) = 10/3 + 30/3 = 40/3 which is a 13.33% return.*

Q3. (15 points) Consider the following probability table below:

| | *X* = 0 | *X* = 1 |
|---|---|---|
| *Y* = 0 | 0.20 | 0.30 |
| *Y* = 1 | 0.10 | 0.40 |

(a) Find the expected value of *X*: *E*(*X*).

*E(X) = Σxp(x) = 0(0.30) + 1 (0.70) = 0.70*

*(b) Find the expected value of X given that Y = 1: E(X|Y=1).*

*E(X|Y = 1) = Σxp(x|y=1) = 0(0.20) + 1(0.80) = 0.80.*

*(c) Find the variance of X: Var(X).*

$Var(X) = \Sigma(x-\mu)^2 p(x) = (0-0.7)^2(0.3) + (1-0.7)^2(0.7) = 0.49(0.3) + 0.09(0.7) = 0.21$

Q4. (15 points) Fred and Barney are in Rock Vegas living life in the fast lane. They are at a casino table where the object of the game is to roll a die. A winning roll is one in which one dot appears. Suppose that the die is fair.

(a) What is the probability that it takes 6 rolls before they get their first winning roll? (Hint: What must be true of the first five rolls, and then that of the sixth?)

*They must get anything but a 1 in their first five rolls. So the probability at each roll is 5/6. On their sixth roll they get a one, which happens with probability 1/6. Thus*

*P(first 1 on sixth roll) =* $(5/6)^5(1/6)$.

(b) What is the probability that in 10 rolls that they get exactly 3 winning rolls? (You need not simplify your answer.)

*Use the binomial formula:*

$$C_3^{10}(1/6)^3(5/6)^{10-3} = \frac{10!}{3!(10-3)!}(1/6)^3(5/6)^7 = \frac{10(9)(8)}{3(2)(1)}(1/6)^3(5/6)^7$$

(c) If the house pays out \$6 for a winning roll, what ante must Fred and Barney put down to make it a fair game?

*The ante must equal the expected payoff: E(payoff) =* $\$6\times(1/6) = \$1$.

Q5. (15 points) Adult North American male weight is (approximately) distributed normal with mean 80 kilograms and variance 100 kilograms$^2$. That is, $W \sim N(80,10^2)$. An elevator has a capacity of 729 kg before the cable snaps.

(a) If 9 random people get on the elevator, what is the probability that they will "go for a ride"? (Hint: What must be true of the sample mean weight for the cable to snap?)

*The cable snaps if the average weight exceeds 81kg. Thus we want to find*

$$P(\bar{X} > 81) = P\left(\frac{\bar{X}-80}{10/\sqrt{9}} > \frac{81-80}{10/\sqrt{9}}\right) = P(Z > 3/10) = 0.3821.$$

(b) What does the central limit theorem (CLT) say about the distribution of the sample mean of weight, even if we know nothing about the distribution of weight?

*The CLT says that the sample mean* $\bar{X}$ *is distributed normal with mean* $\mu$ *and variance* $\sigma^2/n$ *if X is from a normal population. However, even if we know nothing about the population from which it is drawn, the CLT says that for a sufficiently large sample size we know that the sample mean is approximately normal with mean* $\mu$ *and variance* $\sigma^2/n$. *That is, for any X,* $\bar{X} \sim N(\mu,\sigma^2)$ *for n sufficiently large.*

(c) If the population parameters of weight are unknown, and thus estimated with sample statistics, the underlying statistics have a student *t*-distribution. What happens to the *t*-distribution as the sample size increases?

*As the sample size increase the t-distribution approaches that of the standard normal. The tails thin out and for sufficiently large n (over 30) the t-distribution is looks very much like that of the normal.*

Q6. (5 points) Suppose that the arrival time of a bus at a bus stop is uniformly distributed over a 30 minute range: $X \sim U[0,30]$. You arrive at the bus stop. What is the probability that a bus will come in the next 5 minutes?

*Since it is uniform the probability is proportional to the width of the interval.*
$P(0 < X < 5) = 5/30 = 1/6$.

Q7. (5 points) The article "Filling the World's Belly" (The Economist, 2003) discussed the growing problem of obesity. Obesity is defined through a body mass index (BMI) that takes a person's weight in kilograms, and divides it by height, in metres, squared; i.e., BMI = $kg/m^2$. A person is said to be obese if their BMI is over 30. In the United States, 32 percent of adults are considered obese. In Canada the percent of obese adults is 12 percent. Knowing this is it fair to say that there are many more fat people in Canada than in the United States? Explain.

*We cannot conclusively say that American society is much fatter than that of Canada. This is because obese is an ordinal data. It may very well be the case that there are many people in Canada with BMI values of 29.9 and that in the United States that there are many people with BMI values of 30.1.*

Q8. (5 points) In the article "Where Has All the Money Gone?" (The Milken Institute) the author argues that the average American is not much richer today than in the beginning of the 90s in spite of the fact that mean income has increased significantly. What was his argument in making this assertion?

*The difference is accounted by the fact that average income has risen sharply but not median income. This arises because the very rich (the right tail of the income distribution) are getting much richer and hence pulling up the average. Think of it this way, if Bill Gates waltzes inside our classroom the average wealth of people in the class becomes $2 billion. Would it then be fair to say that the average student in our class is richer than the average board member of ABX Gold?*

Q9. (5 points) The article "The War of the Headscarves" (The Economist, 2004) discussed the issue of headscarves and other conspicuous forms of religious attire being banned in public schools in France. A poll conducted by *CSA/Le Parisien* asked a sample of people, "Are you in favour of, or opposed to, a law banning signs or dress that conspicuously display religious affiliation?" The result is listed below:

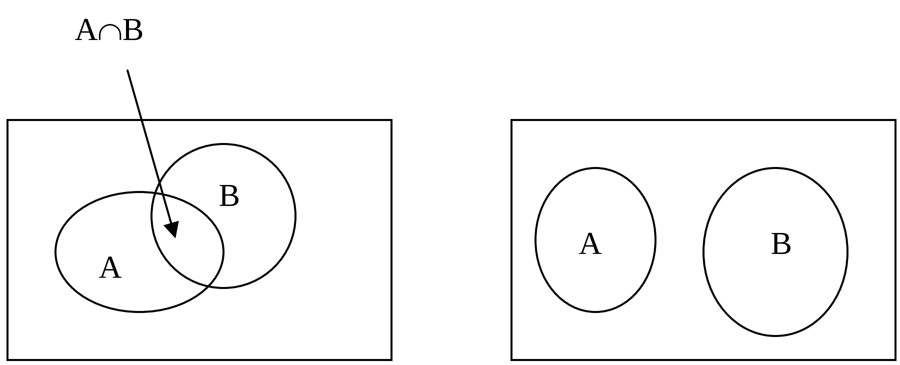
| | In favour, % | Opposed, % |
|---|---|---|
| All French | 69 | 29 |
| Left | 66 | 33 |
| Right | 75 | 24 |
| Muslims | 42 | 53 |
| Muslim women | 49 | 43 |

What percent of Muslim men are opposed to the new law? (You may assume that an equal number of male and female Muslims were surveyed.)

*If there are equal numbers of men and women then 0.5(43) + 0.5(X) = 53. Solving this simple equation we get 63. That is, 63 percent of Muslim men oppose the ban.*

Q10. (5 points) Use a Venn diagram to argue that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

*Just draw a Venn diagram with two events A and B.*




*From the above diagrams we know that we need to subtract $P(A \cap B)$ to avoid double counting the probability of the union of events A and B (which is defined as either A happening or B happening, or both).*

*************************************************************************

Bonus. (10 points) Consider two mutual funds, “Wall Street Boyz” and “Lombardi Street Girlz”. You have $W$ dollars to invest in these two funds. Your wealth is described by the function $W = \alpha X + (1-\alpha)Y$. Where $\alpha X$ is share of wealth in “Wall Street Boyz” and $(1-\alpha)Y$ is share of wealth placed in “Lombardi Street Girlz”, and $\alpha \in [0,1]$. You are a very conservative investor, and your only objective is to minimise the variance of your portfolio. Suppose that:

(i) $\text{Var}(X) = 1$; (ii) $\text{Var}(Y) = 2$; (iii) $\text{Cov}(X,Y) = -1$.

Find the value of $\alpha$ that minimises the variance of your portfolio.

*The variance of this portfolio is*

$$Var(W) = Var[\alpha X + (1-\alpha)Y] = \alpha^2 Var(X) + (1-\alpha)^2 Var(Y) + 2\alpha(1-\alpha)Cov(X,Y)$$
$$= \alpha^2 1 + (1 - 2\alpha + a^2)2 + 2\alpha(1-\alpha)(-1)$$
$$= 5\alpha^2 - 6\alpha + 2$$

*To find the value of $\alpha$ that minimises this function we take its derivative w.r.t. $\alpha$ and solve the equation for zero. Whence $10\alpha - 6 = 0$, and so we have $\alpha = 3/5$. We should technically check the second order condition to see that we have indeed found a minimum.*

<u>Super Bonus</u>. (5 points) In the movie "A Beautiful Mind", starring Russell Crowe as the eccentric Princeton mathematician John Nash, there was a scene in the movie in which Crowe's character tries to explain to his friends the idea of game theory. In particular, he described a strategy in which he and his graduate student colleagues would be able to find themselves dates in the scenario when a beautiful girl along with some of her so-so friends walks into a bar. Why were most economists furious (or at least exasperated) at the analogy used in the movie to describe a "Nash equilibrium"?

*The game he described was not a Nash equilibrium For those of you who watched the movie, recall that he presented the scenario when a beautiful women comes in to a bar with some of her so-so friends. Crowe remarked that it would be a silly strategy for all of them (him and his graduate student colleagues) to pursue the beautiful woman at the cost of neglecting the so-so girls. The beautiful girl would be overwhelmed by the attention and would also feel bad for her friends, who are not receiving any attention, and so in such a scenario the beautiful girl would decline all her suitors, and everyone goes home lonely. Instead, he proposed that none of them pursue the beautiful girl, and instead that they all go for one of the so-so girls. In this way the so-so girls would each be paired up with a courting graduate student and all would be happy. However, if you go on to study game theory, you will realise that the strategy as described above is not an equilibrium.*

*<u>Please do not forget to place this test paper in your exam booklet</u>*

# Introductory Statistics
# Winter 2004

## MIDTERM EXAM

March 2, 2004

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Q1. (15 points) The makers of "Ginsu 2000" (kitchen knives) have decided to market their product through the medium of infomercials. A call centre is set up to handle incoming calls from wannabe sashimi samurais. Suppose that the number of phone calls coming in to the call centre (per hour) is described by a Poisson distribution with parameter $\mu = 9$. That is

$$p(x) = \frac{e^{-9}9^x}{x!} \quad x = 0,1,2,3,\ldots$$

(d) What is the probability that the call centre will receive at most one phone call in the next hour?

*Getting at most one call means getting either 1 or no calls: $P(X \le 1) = p(0) + p(1)$. Thus*

$$P(X \le 1) = p(0) + p(1) = \frac{e^{-9}9^0}{0!} + \frac{e^{-9}9^1}{1!} = e^{-9} + 9e^{-9}$$

(e) What is the probability that the call centre will receive at least four calls in the next hour?

*Use the complement rule:*

$$P(X \ge 4) = 1 - P(X < 4) = 1 - \left[p(0) + p(1) + p(2) + p(3)\right]$$

(f) Using Chebyshev's theorem, find an upper bound for the probability that the call centre will receive 15 or more calls in the next hour. (Hint: If $X$ is distributed Poisson with parameter $\mu$, then $\mathrm{Var}(X) = \mu$.)

*The variance is 9 so the standard deviation is 3. By Chebyshev's theorem we know that at least $1–1/2^2$ of the observations are contained with 2 standard deviations of the mean (9). So at most 1/4 of the observations lie beyond 2 standard deviations of the mean. Moreover, the true number is even smaller since there are two sides of the mean, and we are looking at the right tail.*

Q2. (15 points) Fred and Barney are in Rock Vegas living life in the fast lane. They are at a casino table playing a card game. They are dealt five cards from the deck (without replacement). There are thirteen cards of each suit in a deck of cards (ignore jokers), and assume that the cards are sufficiently shuffled.

(d) What is the probability that they get five cards that are all hearts?

*Since there is no replacement the odds of getting five hearts is*

$$P(5H) = P(H_5 \mid H_4 \mid H_3 \mid H_2 \mid H_1) = (13/52)(12/51)(11/50)(10/49)(9/48)$$

(e) Given that the first three cards they are dealt are aces, what is the probability that they will end up with a four-of-a-kind of aces?

*They have two chances to get an ace in the remaining (49) cards. Thus*

*P(4-of-a-kind|3 Aces) = P(Ace in $4^{th}$ or $5^{th}$ draw) = 1/49 + 1/48.*

*Think of it this way. What is the probability of getting the final ace on the $4^{th}$ draw? It's simply 1/49. If he doesn't get an ace on the $4^{th}$ draw then there are 48 cards remaining and the probability of drawing an ace on his final draw is then 1/48.*

(f) Down \$100 at the end of the night, Fred and Barney try to get out of their debt by playing "double or nothing" on a draw of a card from the deck. If the card suit is spades they owe nothing. Otherwise they owe \$200. What is the size of their expected debt from this gamble?

*With probability 0.75 they owe \$200 and with probability 0.25 they owe nothing. So their expected debt is \$150.*

Q3. (15 points) The finishing time of the Toronto Marathon ($X$) is distributed (approximately) normal with mean 275 minutes and standard deviation 60 minutes: $X \sim N(275,60^2)$.

(a) Find the probability that a runner finishes the race in exactly 3 hours.

*This is zero since we are looking at a continuous variable.*

(b) Find the probability that a runner completes the race in less than 4 hours.

$$P(X < 240) = P\left(\frac{X-275}{60} < \frac{240-275}{60}\right) = P(Z < -5/12) = 0.5 - 0.1628 = 0.3372.$$

(c) You survey 49 random runners. What is the probability that the average finishing time of this group falls in the range of 270 to 280 minutes?

$$P(270 < X < 280) = P\left(\frac{270-275}{60} < \frac{X-275}{60} < \frac{280-275}{60}\right) = P(-1/12 < Z < 1/12) =$$

$$2P(0 < Z < 0.083) = 2(0.033) = 0.066.$$

Q4. (15 points) In the kingdom of Kainada live 5 people – call them "*A*", "*B*", "*C*", "*D*", and "*E*". The incomes of *A*, *B*, *C*, *D*, and *E* are \$5, \$10, \$15, \$20, and \$25, respectively.

(a) Find the median and mean incomes in Kainada.

*Median = \$15. Mean = \$(5 + 10 + 15 + 20 +25)/5 = \$15.*

(b) Find the standard deviation of income in Kainada.

*The variance is $(5-15)^2 + (10-15)^2 + (15-15)^2 + (20-15)^2 + (25-15)^2 = 100 + 25 + 0 + 25 + 100 = 250$. Therefore the standard deviation is $250^{1/2}$, which is approximately 16.*

(c) Find the median and mean incomes if *E*'s income increases to $100.

*The median is unchanged at &15. The mean, however, jumps up to $30.*

Q5. (15 points) Consider the following probability table below:

| | $X = 0$ | $X = 1$ |
|---|---|---|
| $Y = 0$ | 0.30 | 0.10 |
| $Y = 1$ | 0.10 | 0.50 |

(d) Find the probability that $X+Y = 1$: $P(X+Y = 1)$.

*X + Y = 1 when one of the following two scenarios is true: X = 1 and Y = 0, or when X = 0 and Y = 1. The first case happens w.p. 0.10. The second case happens w.p. 0.10. Thus*

$$P(X+Y=1) = P(X=1, Y=0) + P(X=0, Y=1) = 0.20.$$

(e) Find the expected value of X given that $Y = 1$: $E(X|Y=1)$.

*If Y = 1 the w.p.1/6 X = 0, and w.p.5/6 X = 1. Thus E(X|Y=1) = 5/6.*

(f) Find the variance of $X$: Var($X$).

There are two approaches. The simple method is to find E(X) = 0(0.4) + 1(0.6) = 0.6. The find $E(X^2) = 0^2(0.4) + 1^2(0.6) = 0.6$. The use the formula $Var(X) = E(X^2) – [E(X)]^2$. This is then 0.6 – 0.36. Alternatively, you could have found $Var(X) = \Sigma(x-E(X))^2p(x)$. Whence $Var(X) = (0 – 0.6)^2(0.4) + (1 – 0.6)^2(0.6)$ .

Q6. (5 points) In the article "Filling the World's Belly" (The Economist, 2003) it is noted that 1/3 of adult Americans are obese. Thus in a room of ten random adult Americans, what is the probability that there are exactly five obese persons?

*This is a binomial problem. Thus applying the binomial formula with n = 10, x = 5 and p = 1/3, we get*

$$C_5^{10}(1/3)^5(2/3)^5 = \frac{10!}{5!5!}(2/9)^5$$

*which is* $252(32)/9^5$.

Q7. (5 points) State the central limit theorem and explain (in one or two sentences) its significance.

*The CLT says that if X is drawn from a normal population* $X \sim N(\mu,\sigma^2)$ *then the sample mean (* $\overline{X}$ *) has a normal distribution with the same mean, but variance* $\sigma^2/n$*, where n is the sample size. Moreover, even if we do not know anything about the population, so long* *as the sample size is large enough, the sample mean is distributed approximately normal with mean $\mu$ and variance $\sigma^2/n$.*

Q8. (5 points) Suppose that events *A* and *B* are mutually exclusive. Does this imply that they are independent? Explain.

*No. In fact, if two events are mutually exclusive they are "very dependent." In particular, if A and B are independent P(A|B) = P(A). But if they are mutually exclusive P(A|B) = 0. If they are mutually exclusive this means that both cannot happen at the same time, so if B happens then A cannot.*

Q9. (5 points) In the article "Does the US Budget Deficit Matter?" (BBC News, 2004) the report notes that the size of the US budget deficit is at a historically high level. Should this be cause for concern? Explain? Moreover, if projections of future deficits are uncertain how should this affect the mandate at present to rein in spending?

*Just as much as it is stupid for business writers to harp about a triple digit drop in the DJIA when its current level is over 10,000, what matters is not the absolute size of the deficit, as so much as its relative level (i.e., in constant dollars, and as a fraction of GDP). And if projections into the future are uncertain (i.e., a high variance in future predicted values) this would be an impetus for the current administration to be cautious in its spending (assume that as a society we are risk averse – which seems very true given the size of the insurance market, etc.).*

Q10. (5 points) Suppose that the arrival time of a bus at a bus stop is uniformly distributed over a 60 minute range: $X \sim U[0,60]$. You arrive at the bus stop. What is the probability that a bus will come in the next 15 minutes?

*Since the arrival time is uniform the probability of arrival is proportional to its interval.* $P(0 < X < 15) = 15/60 = 1/4$.

************************************************************************

Bonus. (10 points) Consider two mutual funds, "Wall Street Boyz" and "Lombardi Street Girlz". You have *W* dollars to invest in these two funds. Your wealth is described by the function $W = \alpha X + (1-\alpha)Y$. Where $\alpha X$ is share of wealth in "Wall Street Boyz" and $(1-\alpha)Y$ is share of wealth placed in "Lombardi Street Girlz", and $\alpha \in [0,1]$. You are a very conservative investor, and your only objective is to minimise the variance of your portfolio. Suppose that:

(ii) $\text{Var}(X) = 1$; (ii) $\text{Var}(Y) = 2$; (iii) $\text{Cov}(X,Y) = -1$.

Find the value of $\alpha$ that minimises the variance of your portfolio.

*The variance of this portfolio is*

$$Var(W) = Var[\alpha X + (1-\alpha)Y] = \alpha^2 Var(X) + (1-\alpha)^2 Var(Y) + 2\alpha(1-\alpha)Cov(X,Y)$$
$$= \alpha^2 1 + (1 - 2\alpha + a^2)2 + 2\alpha(1-\alpha)(-1)$$
$$= 5\alpha^2 - 6\alpha + 2$$

*To find the value of $\alpha$ that minimises this function we take its derivative w.r.t. $\alpha$ and solve the equation for zero. Whence $10\alpha - 6 = 0$, and so we have $\alpha = 3/5$. We should technically check the second order condition to see that we have indeed found a minimum.*

<u>Super Bonus</u>. (5 points) Kazushi Sakuraba is a well renowned mixed martial arts fighter. He gained his fame by facing and defeating four members of the legendary Gracie family (at different times, of course!). In which dojo does Mr. Sakuraba train? And what is his professed style of fighting?

*Mr. Sakuraba trains at Takada Dojo (along with stable mate Daijiro Matsui). Although Sakuraba professes "pro wrestling" as his fighting style, he is more properly a jiu-jitsu fighter. In any case, his pro wrestling background accounts for his flashy showmanship when he enters the ring.*

*<u>Please do not forget to place this test paper in your exam booklet.</u>*

# Introductory Statistics

**SUGGESTED SOLUTIONS TO MIDTERM EXAM**

March 2, 2004

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*<u>When you are done you must place this test paper in your exam booklet!</u>*

Q1. (15 points) Consider two mutual funds, "Sound Factory Mutual" ($X$) and "Vinyl Funds" ($Y$), whose returns are both normally distributed: $X \sim N(9,5^2)$ and $Y \sim N(12,8^2)$.

(d) What is the probability that Sound Factory Mutual earns at least 10 percent? Likewise, what is the probability that Vinyl Funds earns at least 10 percent?

*We seek $P(X > 10)$ and $P(Y > 10)$. Converting both to statements about Z we get*

$$P\left(\frac{X-9}{5} > \frac{10-9}{5}\right) = P(Z > 0.2) \text{ and}$$

$$P\left(\frac{Y-12}{8} > \frac{10-12}{8}\right) = P(Z > -0.25)$$

*Looking up these values in the table (and a bit of manipulation),we get 0.5–0.0793 = 0.4217 and 0.5 + 0.0987 = 0.5987, respectively.*

(e) What is the probability that Sound Factory Mutual will earn between 0 and 9 percent?

*We seek P(0 < X < 9). Converting this to a statement about Z we get*

$$P\left(\frac{0-9}{5} < \frac{X-9}{5} < \frac{9-9}{5}\right) = P(-1.8 < Z < 0)$$

*Looking at the standard normal table we get 0.4641.*

(f) If an investor placed 1/4 of her money in Sound Factory Mutual and the remainder of her money in Vinyl Funds, what is the expected return on her investment?

*Her expected return is E(W) = (1/4)E(X) + (3/4)E(Y) = 0.25(9) + 0.75(12) = 11.25 %.*

Q2. (15 points) Over the course of 30 days the makers of Weight Gain 4000 ("Je veux être un gâteau à la viande!") have received 480 calls from interested body builders. Let *X* be the number of calls per day to the call centre handling the orders for WG4000, and suppose that *X* follows a Poisson distribution.

$$p(x) = \frac{e^{-\mu}\mu^x}{x!} \quad x = 0,1,2,3,\ldots$$

(g) Find the value of $\mu$ for this question.

*The parameter $\mu$ is just the average "success" rate. That is, how many calls the centre gets in a day (on average). This is simply 480/30 = 16.*

(h) What is the probability that the call centre will receive at least four calls in a given day? (You need not simplify your answer.)

*We seek $P(X \geq 4) = 1 - P(X < 4) = 1 - [p(0) + p(1) + p(2) + p(3)]$ where $p(i) = \frac{e^{-\mu}\mu^i}{i!}$.*

(i) Using Chebyshev's theorem, find an upper bound for the probability that the call centre will receive 24 or more calls in the next hour. (Hint: If *X* is distributed Poisson with parameter $\mu$, then Var(*X*) = $\mu$.)

*Chebyshev's theorem says that at least $1–1/2^2$ of the observations lie within 2 standard deviations of the mean. (Here the mean is 16 and the standard deviation is 4.) So at most 1/4 of the observations can lie beyond X = 24. Moreover, as we are looking only at the right end of the tail, the number should be even smaller.*

Q3. (15 points) Consider the following probability table below:

| | $X = 0$ | $X = 1$ | $X = 2$ |
|---|---|---|---|
| $Y = 0$ | 0.20 | 0.15 | 0.10 |
| $Y = 1$ | 0.10 | 0.20 | 0.25 |

(g) Find the expected value of *X*: E(*X*).

<u>Apply the simple formula $E(X) = \Sigma xp(x) = 0(0.30) + 1(0.35) + 2(0.35) = 1.05$</u>

(h) Find the expected value of *X* given that *Y* = 1: $E(X|Y=1)$.

*We need to rescale the probabilities in the lower box to 10/55, 20/55, and 25/55, respectively. Whence $E(X|Y=1) = 0(10/55) + 1(20/55) + 2(25/55) = 1.27$.*

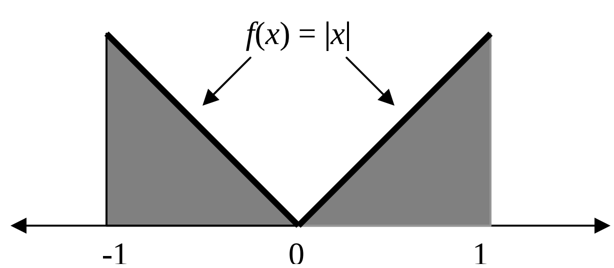
(i) Find $E(X^2)$.

*This is exactly like part (a) except we are looking at $X^2$ instead of X. Then $E(X^2) = 0^2(0.30) + 1^2(0.35) + 2^2(0.35) = 1.75$.*

Q4. (15 points) Suppose that the you have a probability density function given by $f(x) = |x|$ over the interval [-1,1], where $|x|$ is the absolute value of *x*: $|x| = \{x \text{ if } x \geq 0; -x \text{ if } x < 0\}$

a. Find E(*X*).

*Using the fulcrum analogy we get E(X) = 0.*




b. Verify that $f(x) = |x|$, for $–1 \leq x \leq 1$, is indeed a proper pdf (two conditions).

*The two conditions that we need to check are: (i) that the pdf is non-negative over [-1,1], and (ii) the area under the curve is equal to 1. Since $f(x) \geq 0$ for all x, then (i) is clearly satisfied. For (ii), note that the areas are represented by two triangles (of equal area).*

*Each triangle has height 1 and base 1, so each area is $0.5 \times 1 \times 1 = 0.5$. With two triangles the total area is 1.*

c. Find P($-0.5 < X \leq 0.5$)

*Probability is simply the area under a pdf curve. Consider that the triangle on the right (it is symmetric for the left triangle) has base 0.5 and height 0.5. Therefore its area is $(0.5)^3 = 0.125$. And so the total area is 0.25.*

Q5. (15 points) Suppose that the grades for an examination are (approximately) distributed normal with mean 60 and variance 49. That is, $X \sim N(60,7^2)$.

a. The professor wants to raise the class average to 80 by multiplying all test scores by a factor of 4/3. What happens to the variance under this procedure?

*Recall if c is a constant and X a rv, then $Var(cX) = c^2 Var(X)$. Thus the variance increases by a factor of 16/9.*

b. What if instead she raised the grades by adding a factor of 20 to each test score? That is, how does the variance change under this procedure?

*Recall if c is a constant and X a rv, then $Var(c + X) = Var(X)$. That is, there is no change in variance with this procedure.*

c. With the unadjusted test scores, what cut offs need she make if she wishes to flunk (F) 10 percent of the class?

*Looking at the normal table we note that $Z_{0.10} = 1.285$. Thus $-Z_{0.10} = -1.285$, and so the cut off for F should be $60-1.285(7) = 51$.*

Q6. (5 points) Suppose that the arrival time of a bus at a bus stop is uniformly distributed over a 15 minute range: $X \sim U[0,15]$. You arrive at the bus stop. What is the probability that a bus will come in the next 15 minutes?

*Probability is area under a pdf curve. $P(X < 15) = 1$.*

Q7. (5 points) The article "Filling the World's Belly" (The Economist, 2003) discussed the growing problem of obesity. In the United States, 32 percent of adults are considered obese. In Canada the percent of obese adults is 12 percent. There are about 300 million Americans and 30 million Canadians in North America. You meet an obese North American on an airplane. What is the probability that this person is American?

*Let A be the event of being American, and B be the event of being obese. Then apply Bayes' Law:*

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^C)P(A^C)}$$

$$= \frac{0.32(10/11)}{0.32(10/11) + 0.12(1/11)} = 0.9639$$

*Where* $A^C$ *is the event of being Canadian. And P(A) = 300/330 and* $P(A^C)$ *= 30/330.*

Q8. (5 points) The article "The War of the Headscarves" (The Economist, 2004) discussed the issue of headscarves and other conspicuous forms of religious attire being banned. A poll conducted by *CSA/Le Parisien* asked a sample of people, "Are you in favour of, or opposed to, a law banning signs or dress that conspicuously display religious affiliation?" The result is listed below:

| | In favour, % | Opposed, % |
|---|---|---|
| All French | 69 | 29 |
| Left | 66 | 33 |
| Right | 75 | 24 |
| Muslims | 42 | 53 |
| Muslim women | 49 | 43 |

What percent of Muslim men are in favour of the new law? (Assume that an equal number of male and female Muslims were surveyed.)

*We know that 49(0.5) + X(0.5) = 42* $\Rightarrow$ *X = 35. That is, 35% of Muslim men are in favour of the proposed law.*

Q9. (5 points) State the central limit theorem (CLT).

*The central limit theorem says: "The sampling distribution of the mean of a random sample drawn from any population is approximately normal for a sufficiently large sample size." (pg. 274, Keller) That is, if* $X \sim (\mu,\sigma^2)$ *then* $\overline{X} \sim N(\mu,\sigma^2/n)$ *for n sufficiently large (and the relationship is exact if* $X \sim N(\mu,\sigma^2)$*).*

Q10. (5 points) In the article "Where Has All the Money Gone?" (The Milken Institute), the author argues that the average American is not much richer today than in the beginning of the 90s in spite of the fact that mean income has increased significantly. What was his argument in making this assertion?

*The difference is accounted by the fact that average income has risen sharply but not median income. This arises because the very rich (the right tail of the income distribution) are getting much richer and hence pulling up the average. Think of it this way, if Bill Gates waltzes inside our classroom the average wealth of people in the class becomes $2 billion. Would it then be fair to say that the average student in our class is richer than the average board member of ABX Gold?*

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Bonus. (10 points) Consider two mutual funds, "Wall Street Boyz" and "Lombardi Street Girlz". You have $W$ dollars to invest in these two funds. Your wealth is described by the function $W = \alpha X + (1-\alpha)Y$. Where $\alpha X$ is share of wealth in "Wall Street Boyz" and $(1-\alpha)Y$ is share of wealth placed in "Lombardi Street Girlz", and $\alpha \in [0,1]$. You are a very conservative investor, and your only objective is to minimise the variance of your portfolio. Suppose that:

(i) $Var(X) = 1$; (ii) $Var(Y) = 2$; (iii) $Cov(X,Y) = -1$.

Find the value of $\alpha$ that minimises the variance of your portfolio.

*The variance of this portfolio is*

$$
\begin{aligned}
&Var(W) = Var[\alpha X + (1-\alpha)Y] = \alpha^2 Var(X) + (1-\alpha)^2 Var(Y) + 2\alpha(1-\alpha)Cov(X,Y) \\
&= \alpha^2 1 + (1 - 2\alpha + a^2)2 + 2\alpha(1-\alpha)(-1) \\
&= 5\alpha^2 - 6\alpha + 2
\end{aligned}
$$

*To find the value of $\alpha$ that minimises this function we take its derivative w.r.t. $\alpha$ and solve the equation for zero. Whence $10\alpha - 6 = 0$, and so we have $\alpha = 3/5$. We should technically check the second order condition to see that we have indeed found a minimum.*

<u>Super Bonus</u>. (5 points) Hannah Barbara Productions originally had intended that Fred and Wilma, of the "Flintstones" cartoon, were to have a baby boy. As we all know, in the end they decided on a having a female baby, "Pebbles" (and in my opinion it was the best damn decision they ever made). What name was proposed for the male baby that never came to fruition?

*Hannah Barbara had originally planned on producing a baby boy, whose shtick was supposed to have been that he was "a chip off the old block". His name was supposed to have been Fred Jr. As you can see, Pebbles was a much better choice.*

*<u>Please do not forget to place this test paper in your exam booklet.</u>*