MPP Calculus
Take-Home Exam Solutions

1 Continuity

- Suppose that f assumes only rational values over an interval; moreover, it assumes two distinct values over the interval. Prove that f must be a discontinuous function.

Proof. Suppose that f is continuous. Then by the intermediate value theorem (IVT) f must assume all the values in between the two distinct rational numbers. Since the irrational numbers are dense (i.e., there exists an irrational number between any two distinct rational numbers) this means that f would have to assume irrational values, too. ■

- Find the points of discontinuity of the following functions:

1.
$$f(x) = \frac{1 - \log x}{1 + \log x}$$

First note that f is defined only for $x > 0$. Now points of discontinuity are precisely when the denominator is zero.

$$\begin{aligned} 1 + \log x &= 0 && (1) \\ &\Rightarrow x = e^{-1} > 0. && (2) \end{aligned}$$

2.
$$f(x) = \frac{1 + e^{-x}}{e^{x} + \log(1 + x)}$$

First note that f is defined only for $x > -1$. Now points of discontinuity occur when the denominator is zero.

$$\begin{aligned} e^{x} + \log(1 + x) &= 0 && (3) \\ &\Rightarrow \log(1 + x) = -e^{x} && (4) \end{aligned}$$

Although this proves to be hard to explicitly solve we can guess a range for the value of x that satisfies this equation. Since e^x is always positive then $-e^x$ is always negative. Now note that $\log y$ is negative for $y \in (0, 1)$. Thus conclude that $x \in (-1, 0)$.

3.
$$f(x) = \frac{3x^2 + \int t \log t dt}{(1 + x) \int_a^x \frac{1}{t} dt}$$

Note that the definition of

$$\int_a^x \frac{1}{t} dt = \log x - \log a. \qquad (5)$$

Thus f is defined only for $x > 0$. Moreover, the denominator is zero when $x = a$.

2 Rolle's Theorem

Suppose that f is defined everywhere and that all its derivatives are continuous, in particular, $f''' < 0$. Show that f has at most two critical points.

Proof. Suppose that f has three (or more) distinct critical points. That is, suppose there exist x_1, x_2, and x_3 such that

$$f'(x_1) = f'(x_2) = f'(x_3) = 0. \tag{6}$$

By Rolle's theorem this implies that there exist $x_4 \in (x_1, x_2)$ and $x_5 \in (x_2, x_3)$ such that

$$f''(x_4) = f''(x_5) = 0. \tag{7}$$

Now applying Rolle's theorem once again, this implies that there exists $x_6 \in (x_4, x_5)$ such that

$$f'''(x_6) = 0.$$

■

3 Graphing

Sketch the graphs of the following two functions and explicitly label all the points of interest (minima, maxima, points of inflection, roots, etc.):

$$f(x) = \frac{x}{1+x} \tag{8}$$

and

$$f(x) = \frac{e^{-x}}{1+e^x}. \tag{9}$$

For the first function note that f is undefined at the point $x = -1$. Then

$$\lim_{x \to -1^-} f(x) = \infty \quad \text{and} \quad \lim_{x \to -1^+} f(x) = -\infty.$$

Furthermore, f has a horizontal asymptote

$$\lim_{x \to \infty} f(x) = 1 = \lim_{x \to -\infty} f(x).$$

Now note that

$$f'(x) = \frac{1}{(1+x)^2} > 0 \tag{10}$$

and

$$f''(x) = -\frac{2}{(1+x)^3}. \tag{11}$$

The first derivative being positive tells us that the graph is everywhere increasing. The second derivative is less than or greater than zero as x is greater than or less than -1.

For the second function note that it is everywhere positive since $e^x > 0$ for all x. The first derivative of the function is

$$f'(x) = -\frac{2+e^x}{(1+e^x)^2} < 0. \tag{12}$$

Moreover, its second derivative is

$$f''(x) = \frac{4e^x + e^{-x} + 3}{(1+e^x)^3} > 0. \tag{13}$$

Thus the graph of f is everywhere decreasing and convex.

4 Inverse Functions

Find the inverse functions for the following two functions:

$$f(x) = \sqrt[3]{1 + \log x} \tag{14}$$

and

$$f(x) = \frac{1}{x}. \tag{15}$$

Also, be sure to state the domains for f and f^{-1}.

The first function is defined only for $x > 0$. To find its inverse we set

$$x = \sqrt[3]{1 + \log y} \;\Rightarrow\; y = f^{-1}(x) = \exp\left(x^3 - 1\right). \tag{16}$$

The inverse function is defined for all x. The inverse is well defined since f is $1 - 1$.

For the second function note that we have a point of discontinuity at $x = 0$. Note that $f' < 0$ for all $x \neq 0$, and so the inverse is well defined and exists for all values of $x \neq 0$. Moreover, $f^{-1}(x) = 1/x$.

5 Differentiation

Find the first and second derivatives of the following two functions:

$$f(x) = \int_{x^2+2}^{0} \sqrt{1 + t^2}dt \tag{17}$$

and

$$f(x) = x\left[1 + (\log(1 + x))^3\right]. \tag{18}$$

This merely tested whether you understood (or could keep track of) the rules of differentiation (and a little bit of FTC). For the first function we rearrange the limits of integration and then apply the FTC.

$$f(x) = -\int_0^{x^2+2} \sqrt{1 + t^2dt} \tag{19}$$

$$\Rightarrow \quad f'(x) = -2x\sqrt{1 + (x^2 + 2)^2}. \tag{20}$$

To find f'' apply the product rule and the chain rule.

$$f''(x) = -2\sqrt{1 + (x^2 + 2)^2} - 4x^2\left(x^2 + 2\right)\left[1 + \left(x^2 + 2\right)^2\right]^{-1/2}. \tag{21}$$

For the second integral

$$f'(x) = 1 + [\log(1 + x)]^3 + \frac{3x}{1 + x}[\log(1 + x)]^2 \tag{22}$$

and

$$f''(x) = \frac{3}{1 + x}[\log(1 + x)]^2 + 2\frac{3x}{(1 + x)^2}\log(1 + x) + \frac{3}{(1 + x)^2}[\log(1 + x)]^2. \tag{23}$$

6 Integration

Solve the following two integrals:

$$\int_0^1 xe^{-x}dx \tag{24}$$

and

$$\int_e^\infty \frac{1}{x\log x}dx. \tag{25}$$

For the first integral use IBP where we let $u = x$ (which implies $du = dx$) and $dv = e^{-x}dx$ (which implies $v = -e^{-x}$). Then using the formula $\int udv = uv - vdu$ we get

$$\begin{aligned} \int_0^1 xe^{-x}dx &= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx && (26)\\ &= -e^{-1} - \left(e^{-1} - 1\right) && (27)\\ &= 1 - 2/e. && (28) \end{aligned}$$

For the second integral we use substitution, where we let $w = \log x$ so that $dw = (1/x)\,dx$. Then

$$\begin{aligned} \int_e^\infty \frac{1}{x\log x}dx &= \int_e^\infty \frac{1}{w}dw && (29)\\ &= \log w \mid_e^\infty = \infty. && (30) \end{aligned}$$

7 Taylor Polynomials

Find the Taylor polynomials of degree 4 for the following functions:

1.
$$f(x) = \frac{1}{1-x} \quad \text{about} \quad a = 2.$$

$$f(x) = \exp(e^x) \quad \text{about} \quad a = 0.$$

This is straightforward. Simply recall the definition

$$P_{4,a}(x) = \sum_{k=0}^{4} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{31}$$

For the first function note that

$$\begin{aligned} f^{(0)}(2) &= f(2) = -1\\ f^{(1)}(2) &= f'(2) = 1\\ f^{(2)}(2) &= -2\\ f^{(3)}(2) &= 6\\ f^{(4)}(2) &= -24. \end{aligned}$$

Thus

$$P_{4,a=2}(x) = -1 + (x-2) - (x-2)^2 + (x-2)^3 - (x-2)^4. \tag{32}$$

For the second function note that

$$\begin{aligned}
f^{(0)}(0) &= f(0) = e \\
f^{(1)}(2) &= f'(2) = e \\
f^{(2)}(2) &= 2e \\
f^{(3)}(2) &= 5e \\
f^{(4)}(2) &= 15e.
\end{aligned}$$

Thus,

$$P_{4,a=0}(x) = e + ex + ex^2 + \frac{5}{6}ex^3 + \frac{15}{24}ex^4 \tag{33}$$

$$= e\left(1 + x + x^2 + \frac{5}{6}x^3 + \frac{15}{24}x^4\right). \tag{34}$$

8 Sequences & Series

- Show that the sequence $\{1 + 1/n^3\}$ converges to 1 directly from the definition of convergence.

Proof. We claim that the sequence $\{1 + 1/n^3\}$ converges to 1 as $n \to \infty$. Then for every $\varepsilon > 0$ we need to show that $|1 + 1/n^3 - 1| < \varepsilon$ whenever $n \geq N$. That is, for any fixed $\varepsilon > 0$ we want $|1/n^3| < \varepsilon$. It is clear that for all $n \geq N = 1/\varepsilon^3$ that this condition will be satisfied. ■

- Find the sum of the sequences (i): $\{1/n\}$; (ii) and $\{1/n^3\}$.

The first sequence is not summable (it is the harmonic series) since $\log x$ is unbounded. For the second sum use the integral

$$\lim_{A\to\infty} \int_1^A \frac{1}{x^3} dx \tag{35}$$

to verify that it is summable and that its sum is 3/2.

- Find the following limit:

$$\lim_{n\to\infty} \left(1 - \frac{3}{n}\right)^n. \tag{36}$$

Use the $x = \exp[\log(x)]$ trick and then apply L'Hôpital's rule.

$$\lim_{n\to\infty} \exp\left[n \log\left(1 - \frac{3}{n}\right)\right] = \exp\left[\lim_{n\to\infty} \frac{\log\left(1 - \frac{3}{n}\right)}{\frac{1}{n}}\right] \tag{37}$$

$$= \exp\left[\lim_{m\to 0} \frac{\log(1 - 3m)}{m}\right] \tag{38}$$

$$= \exp\left[\lim_{m\to 0} \frac{-3}{1 - 3m}\right] \tag{39}$$

$$= e^{-3}. \tag{40}$$

9 Matrix Algebra

Given

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 6 & 3 \\ 5 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Solve the following matrix operations (if they are indeed valid):

1. AB

   Not conformable: $(2 \times 2) \times (3 \times 2)$.

2. $B + C$

   Not conformable

3. BAC

$$BAC = \begin{bmatrix} 7 & 10 & 7 \\ 15 & 24 & 15 \\ 17 & 26 & 17 \end{bmatrix} \tag{41}$$

4. $(BAC)^{-1}$

   The inverse does not exist since columns 1 and 3 are identical.

5. $\det(BAC)$

   Since the inverse does not exist we can conclude that its determinant is zero.

6. BA'

$$BA' = \begin{bmatrix} 5 & 11 \\ 12 & 30 \\ 13 & 31 \end{bmatrix}. \tag{42}$$

   Note that $BA' \neq (BA)' = A'B'$.

- Use Cramer's Rule to solve the following system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 - 2x_2 + 4x_3 &= -2 \end{aligned}$$

This is a straightforward application of Cramer's rule,

$$\overline{x}_j = \frac{\det A_j}{\det A} \tag{43}$$

where A_j is the matrix formed by substituting the d vector into the j^{th} column of the A matrix. Whence

$$\begin{aligned} \overline{x}_1 &= -\frac{48}{29} \\ \overline{x}_2 &= \frac{49}{29} \\ \overline{x}_3 &= \frac{22}{29} \end{aligned}$$

10 Optimisation with Constraints

Solve the following optimisation problems:

1. $$\max_{x_1,x_2} U(x_1,x_2) = x_1^2 x_2^2 \tag{44}$$

   subject to the restriction

   $$x_1 + \frac{3}{2}x_2 = 100. \tag{45}$$

   Although the objective function is not linear we can still use the method of Lagrange multipliers since the constraint is linear and binding, and the fact that the objective is linear in log. Set up a Lagrangian function

   $$L = x_1^2 x_2^2 - \lambda\left(x_1 + \frac{3}{2}x_2 - 100\right) \tag{46}$$

   and then compute the three FOCs:

   $$\begin{aligned} \partial x_1 &: \quad 2x_1x_2^2 - \lambda = 0 & (47)\\ \partial x_2 &: \quad 2x_1^2x_2 - \frac{3}{2}\lambda = 0 & (48)\\ \partial \lambda &: \quad x_1 + \frac{3}{2}x_2 - 100 = 0 & (49) \end{aligned}$$

   Then set

   $$\begin{aligned} 2x_1x_2^2 &= \frac{4}{3}x_1^2x_2 & (50)\\ &\Rightarrow \quad x_1 = \frac{3}{2}x_2. & (51) \end{aligned}$$

   Plug this back into the original constraint whence $x_1^* = 50$ and $x_2^* = 100/3$.

2. $$\min_{K,\ L} rK + wL \tag{52}$$

   subject to the restrictions

   $$K^{1/3}L^{2/3} = 100. \tag{53}$$

   Use the fact that

   $$K^{1/3}L^{2/3} = 100 \ \Leftrightarrow\ \frac{1}{3}\log K + \frac{2}{3}\log L = \log 100. \tag{54}$$

   Then write the Lagrangian

   $$L = -(rK + wL) - \lambda\left(\frac{1}{3}\log K + \frac{2}{3}\log L - \log 100\right). \tag{55}$$

   The three FOCs are

   $$\begin{aligned} \partial K &: \quad -r - \frac{\lambda}{3K} = 0 & (56)\\ \partial L &: \quad -w - \frac{2\lambda}{3L} = 0 & (57)\\ \partial \lambda &: \quad \frac{1}{3}\log K + \frac{2}{3}\log L - \log 100 = 0 & (58) \end{aligned}$$

Then use the fact that

$$-3rK = -\frac{3}{2}wL \;\Rightarrow\; K = \frac{1}{2}\frac{w}{r}L. \tag{59}$$

Plug this back into the constraint

$$\begin{aligned}\frac{1}{3}\left[\log\frac{1}{2} + \log\frac{w}{r} + \log L\right] + \frac{2}{3}\log L &= \log 100 \\ &\Rightarrow \quad \log L = \log 100 + \log\left(\frac{w}{2r}\right)^{-1/3} \\ &\Rightarrow \quad L^* = 100\left(\frac{w}{2r}\right)^{-1/3}.\end{aligned}$$

3. Repeat question 2 with the modification $K^{1/3}L^{2/3} \geq 100$ and $K \geq 0$, $L \geq 1000$.

   The condition $K^{1/3}L^{2/3} \geq 100$ will bind since the production function is locally nonsatiated. Moreover, if given w and r the level of L^* is less than 1000 then the condition $L \geq 1000$ will bind. If L^* is greater than 1000 the condition does not bind and the solution above is adequate.

4.

$$\min_{x_1,x_2} 4x_1 + x_2 \tag{60}$$

subject to the restrictions

$$\begin{aligned} x_1^2 - 4x_1 + x_2 &\geq 1 && (61)\\ -2x_1 - 3x_2 &\geq -11 && (62)\\ x_1,\ x_2 &\geq 0 && (63)\end{aligned}$$

We have to use Kuhn-Tucker to solve this problem. Recall that minimisation is equivalent to maximising the negative of the objective function. Moreover, the maximisation constraints are of the form $g^i(x) \leq c_i$.

$$Z = -(4x_1 + x_2) - \mu_1\left(4x_1 - x_1^2 - x_2 + 1\right) - \mu_2(2x_1 + 3x_2 - 11). \tag{64}$$

The Kuhn-Tucker conditions are

$$\begin{aligned}\frac{\partial Z}{\partial x_j} &\geq 0 \quad x_j \geq 0 \quad \text{and} \quad x_j\frac{\partial Z}{\partial x_j} = 0 \\ \frac{\partial Z}{\partial \mu_i} &\geq 0 \quad \mu_i \geq 0 \quad \text{and} \quad \mu_i\frac{\partial Z}{\partial \mu_i} = 0\end{aligned}$$

If the qualification constraint and the sufficiency theorem are satisfied the above six equations implicitly define the solution to the minimisation problem. Indeed, since this is a two-variable system we can find the solution graphically (or at the very least, illustrate the feasible region of the problem).